Measuring efficiency of a hierarchical organisation with fuzzy DEA method

Măsurarea eficienţei unei organizaţii de tip ierarhic cu metoda DEA fuzzy

Professor Florica LUBAN, Ph.D.
The Bucharest Academy of Economic Studies, Romania
e-mail: florica.luban@man.ase.ro

Abstract

The paper analyses how the data envelopment analysis (DEA) and fuzzy set theory can be used to measure and evaluate the efficiency of a hierarchical system with n decision making units and a coordinating unit. It is presented a model for determining the of activity levels of decision making units so as to achieve both fuzzy objectives of achieving global target levels of coordination unit on the inputs and outputs and individual target levels of decision making units, and then some methods to resolve fuzzy models are proposed.

Keywords: *fuzzy DEA, policy making in multi-level organisations, efficiency analysis*

Rezumat

Lucrarea analizează modul în care metoda data envelopment analysis (DEA) şi teoria mulţimilor fuzzy pot fi utilizate pentru a măsura şi evalua eficienţa unui sistem ierarhic cu n unităţi de luare a deciziilor şi o unitate de coordonare. Este prezentat un model pentru determinarea nivelurilor de activitate ale unităţilor de luare a deciziilor astfel încât să se realizeze atât obiectivele fuzzy de atingeree a nivelurilor ţintă la nivel global ale unităţii coordonatoare referitoare la intrări şi ieşiri cât şi nivelurile ţinţă individuale ale unităţilor de luare a deciziilor şi apoi sunt propuse căteva metode de rezolvare a modelelor fuzzy.

Cuvinte-cheie: *metoda DEA fuzzy, elaborarea deciziilor în organizaţiile multi-nivel, analiza eficienţei*

JEL Classification: C61, C65, D61, L25

Introduction

Organisations such as fast-food chains, hospitals, banks, schools, university departments are organizations with many decision-making units. One of the important frameworks used for efficiency measurement of decision-making units (DMUs) in organizations is the data envelopment analysis (DEA), occasionally called frontier analysis, which was first put forward by Charnes, Cooper and Rhodes (1978). This model abbreviated CCR assumes a constant return to scale and was modified by Banker, Charnes and Cooper (1984) into BBC model to suit for cases of variable returns to scale. Since the pioneering works for DEA, a great variety of models and applications have been reported (Seiford, 1996). In (Charnes et al., 1994; Cooper, Park and Yu, 1999; Cooper, Park and Yu, 2001) it is recognized the need to introduce some kind of data uncertainty into the linear programming models of data envelopment analysis, because the non-parametric frontier models are extremely sensitive to measurement errors and outliers.

To deal quantitatively with imprecision in decision process, Zadeh (1965) and Bellman and Zadeh (1970) have been introduced the notion of fuzziness. In their approach the objectives and constraints are treated as fuzzy sets in the decision space and a fuzzy decision then would be obtained as the intersection of these fuzzy sets (see also (Dumitru and Luban, 1986; Luban, 2003; Zimmermann, 1991).

There are several fuzzy approaches to the assessment of efficiency in the DEA literature. Sengupta (1992) applies principles of fuzzy set theory to DEA. The fuzziness is incorporated into DEA model by defining tolerance levels on both objective function and constraint violations. Triantis and Girod (1998) use membership function values to transform fuzzy input and output data into crisp data and to develop a mathematical programming model. Guo and Tanaka (2001) propose a fuzzy CCR model in which the fuzzy constraints are converted into crisp constraints by predefining a possibility level and using the comparison rule for fuzzy numbers. In (Leon et al., 2003) it is developed a fuzzy BBC model. Lertworasirikul et al. (2003) treat the fuzzy constraints as fuzzy events and develop a possibility DEA model by using possibility measures on fuzzy events.

Entani, Maeda and Tanaka (2002) propose a DEA model with interval efficiencies. Their model was first developed for crisp data and then extended to interval data and fuzzy data. A complete survey of state of the art about fuzzy interval analysis is presented in (Dubois and Prade, 2000). Kao and Liu (2000) propose transforming fuzzy data into interval data by applying the α-level sets (also called α-cuts). This approach is also adopted by Saati and Memariani (2005) so that all DMUs could be evaluated using a common set of weights under a given α-cut.

Wang, Greatbanks and Yang (2005), Zhu (2003) propose interval DEA models for measuring the lower and upper bounds of the best relative efficiency of each DMU with interval and/or fuzzy input-output data.

The organizations with many decision-making units are multi-level organizations. In this context, the decision maker can be faced with several problems of meeting efficiency and effectiveness goals. The first problem is to measure and evaluate efficiency in terms of input and output at the decision-making unit level. The second problem is to relate operational level efficiency to global organizational targets or effectiveness. To address these problems a multi-level programming approach is proposed in the literature. In this context, goal programming approach has been extensively used. Athanassopoulos (1995) proposes o model integrating goal programming and data envelopment analysis (GoDEA) to incorporate target setting and resource allocation in multi-level planning problems. In (Hoopes, Triantis and Partangel, 2000) the GoDEA formulation is used to assess the performance of serial manufacturing technologies found in a two-level hierarchical manufacturing organizations. Narsimhan (1980) proposes a model that integrates the concepts of fuzzy set theory and goal programming.

The generalized goal programming approach seeks to minimize the negative and positive deviations from some given aspiration levels. However, in most real life situations, the aspiration levels need a fuzzy formulation which allows imprecise specification of the aspiration levels. Sheth and Triantis (2003) propose a Fuzzy Goal Data Envelopment Analysis (Fuzzy GoDEA) framework to measure and evaluate the goals of efficiency and effectiveness in a fuzzy environment. A membership function is defined for each fuzzy constraint associated with the efficiency and effectiveness goals.

The rest of the paper is organized as follows. Section 2 provides a short description of CCR model and of its dual. The membership functions associated to fuzzy goals of a model for a hierarchical environment with two levels of decision making are presented in Section 3. In Section 4 some methods for solving fuzzy DEA models are described.

DEA models

The most frequently used DEA model is the CCR model, named after Charnes, Cooper and Rhodes (1978). Suppose that there are *n* DMUs, each of which consumes the same type of inputs and produce the same type of outputs. Let *m* be the number of inputs and let *s* be the number of outputs. All inputs and outputs are assumed to be nonnegative, but at least one input and one output are positive. The following notation will be used:

DMUj the jth DMU
DMU^c the assessed DMU

x_{ij} level of input *i* for DMUj

y_{rj} level of output *r* for DMUj

x_i^c, y_r^c level of input *i* and output *r* when assessing DMU^c

u_r^c, v_i^c weights for output *r* and input *i* when assessing DMU^c

λ_j^c activity level of DMUj when assessing DMU^c

Θ^c efficiency index of DMU^c

In the CCR model, the multiple inputs and multiple outputs of each DMU are aggregated into a single virtual input and a single virtual output, respectively. The CCR model and its dual DCCR can be formulated as the following linear programming models:

$$
\begin{aligned}
&\text{(CCR)}\ \ \text{Max} \sum_{r=1}^{s} u_r^c y_r^c \\
&\text{subject to:} \\
&\sum_{i=1}^{m} v_i^c x_i^c = 1 \\
&-\sum_{i=1}^{m} v_i^c x_{ij} + \sum_{r}^{s} u_r^c y_{rj} \le 0, \quad j = 1, \ldots, n \\
&v_i^c \ge 0, \quad i = 1, \ldots, m \\
&u_r^c \ge 0, \quad r = 1, \ldots, s
\end{aligned} \qquad (1)
$$

$$
\begin{aligned}
&\text{(DCCR)}\ \ \text{Min}\ \Theta^c \\
&\text{subject to:} \\
&\Theta^c x_i^c - \sum_{j=1}^{n} \lambda_j^c x_{ij} \ge 0, \quad i = 1, \ldots, m \\
&\sum_{j=1}^{n} \lambda_j^c y_{rj} \ge y_r^c, \quad r = 1, \ldots, s \\
&\lambda_j^c \ge 0, \quad j = 1, \ldots, n \\
&\Theta^c \in R
\end{aligned} \qquad (2)
$$

Θ^c is the fraction of DMU^c input required by the composite unit.

The objective function for DCCR is to minimize the input resources available to the composite unit.

In the DCCR model, from the constraints $\Theta^c x_i^c - \sum_{j=1}^{n} \lambda_j^c x_{ij} \geq 0$, and $\lambda_j^c \geq 0$ it results the condition $\sum_{j=1}^{n} \lambda_j^c = 1$. The model (2) becomes:

$$\text{Min } \Theta^c$$

subject to:

$$\Theta^c x_i^c - \sum_{j=1}^{n} \lambda_j^c x_{ij} \geq 0, \quad i = 1, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j^c y_{rj} \geq y_r^c, \quad r = 1, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j^c = 1$$

$$\lambda_j^c \geq 0, \quad j = 1, \ldots, n \tag{3}$$

From the duality theorem of linear programming, the optimal values of the CCR and DCCR models are equal. Let Θ^{c*} be the optimal objective value (efficiency value of the assessed DMU). From the constraints of model (1), an efficiency value of the assessed DMU falls in the range of (0, 1]. In (Charnes et al. 1994) it is shown that a DMU is fully efficient if and only if it is impossible to improve any input or output without worsening some other inputs or outputs.

As has been shown in Section 1, there are many ways to expand DEA mathematical programming formulations using fuzzy set theory. In this paper, the focus will be on the DCCR model to develop a Fuzzy DEA model for a hierarchical system.

Membership functions associated to fuzzy goals

In model (3), a set of activity levels is obtained when each DMU is assessed. This activity levels are DMU specific. Thus each DMU when assessed has its own set of activity levels for each input and output for all the DMUs in the data set. These activity levels λ_j^c will be used to develop a Fuzzy DEA model for a hierarchical system.

In (Sheth and Triantis 2003), a model is developed for a hierarchical environment with two levels of decision making. This hierarchical system has *n* DMUs and a coordinating CDMU. The CDMU provides global input targets *TXi* and global output targets *TYr*, and pre-specifies tolerance limits for the global targets. The individual DMUs specify the tolerance limits for their individual inputs and outputs. The aim is to restrict global consumption of each input to less than or equal to the global target *TXi* and to enable global production of output that is more than or equal to the global target *TYr*.

The model for determining the activity levels λ_j^c that maximally achieve the fuzzy goals of meeting global targets and meeting individual DMU targets has the following fuzzy constraints:

Achievement of individual DMU targets:

$$\sum_{j=1}^{n} \lambda_j^c x_{ij} \underset{\sim}{\leq} x_i^c, \quad i = 1, \ldots, m; \quad c = 1, \ldots, n \tag{4}$$

$$\sum_{j=1}^{n} \lambda_j^c y_{rj} \underset{\sim}{\geq} y_r^c, \quad r = 1, \ldots, s; \quad c = 1, \ldots, n \tag{5}$$

Achievement of global targets:

$$\sum_{j=1}^{n} \lambda_j^1 x_{ij} + \ldots + \sum_{j=1}^{n} \lambda_j^n x_{ij} \underset{\sim}{\leq} TXi, \quad i = 1, \ldots, m \tag{6}$$

$$\sum_{j=1}^{n} \lambda_j^1 y_{rj} + \ldots + \sum_{j=1}^{n} \lambda_j^n y_{rj} \underset{\sim}{\geq} TYr, \quad r = 1, \ldots, s \tag{7}$$

$$\sum_{j=1}^{n} \lambda_j^c = 1, \quad c = 1, \ldots, n \tag{8}$$

$$\lambda_j^c \geq 0, \quad j = 1, \ldots, n; \quad c = 1, \ldots, n \tag{9}$$

where $\underset{\sim}{\leq}$, $\underset{\sim}{\geq}$ denote fuzzyfication of the goals or constraints.

The fuzzy goal imply that the goal have to be essentially met within the specified tolerance limits or bounds. These bounds are pre-specified by the decision-maker based on historical knowledge. Let be:

ℓ_r^c lower bound on DMU output target y_r^c

w_i^c upper bound on DMU input target x_i^c

Lr lower bound on global output target TYr and $Lr = \sum_{c=1}^{n} \ell_r^c$

Wi upper bound on global input target TXi and $Wi = \sum_{c=1}^{n} w_r^c$.

The membership functions associated with fuzzy goals (4) – (7) can be expressed as following:

For meeting individual DMU targets:

$$\mu_{x_i^c} = \frac{w_i^c - \sum_{j=1}^{n} \lambda_j^c x_{ij}}{w_i^c - x_i^c}, \quad i = 1, \ldots, m; \quad c = 1, \ldots, n \tag{10}$$

$$\mu_{y_r^c} = \frac{\sum_{j=1}^{n} \lambda_j^c y_{rj} - \ell_r^c}{y_r^c - \ell_r^c}, \quad r = 1, \ldots, s; \quad c = 1, \ldots, n \tag{11}$$

For meeting global targets:

$$\mu_{x_i} = \frac{Wi - (\sum_{j=1}^{n} \lambda_j^1 x_{ij} + \ldots + \sum_{j=1}^{n} \lambda_j^n x_{ij})}{Wi - TXi}, \quad i = 1, \ldots, m \tag{12}$$

$$\mu_{y_r} = \frac{(\sum_{j=1}^{n} \lambda_j^1 y_{rj} + \ldots + \sum_{j=1}^{n} \lambda_j^n y_{rj} - Lr}{TYr - Lr}, \quad r = 1, \ldots, s \tag{13}$$

With the membership functions defined by (10) – (13), some crisp equivalent linear programs can be developed.

Methods for solving fuzzy DEA models

When all membership functions are determined, the following linear programming model (Wu and Guu 2001), (Luban 2003), can be obtained:

Max α
subject to:

$$\alpha \leq \mu_{x_i^c} \leq 1, \quad i = 1, \ldots, m; \quad c = 1, \ldots, n$$
$$\alpha \leq \mu_{y_r^c} \leq 1, \quad r = 1, \ldots, s; \quad c = 1, \ldots, n$$
$$\alpha \leq \mu_{x_i} \leq 1, \quad i = 1, \ldots, m$$
$$\alpha \leq \mu_{y_r} \leq 1, \quad r = 1, \ldots, s$$
$$\alpha \in [0, 1]$$
$$\sum_{j=1}^{n} \lambda_j^c = 1, \quad c = 1, \ldots, n$$
$$\lambda_j^c \geq 0, \quad j = 1, \ldots, n; \quad c = 1, \ldots, n \qquad (14)$$

The optimal value α^* obtained by solving model (14) denotes that the satisfaction level for all membership functions can simultaneously obtain.

Further, let us assume that all membership functions are equally important. The DEA problem can be solved by the following average operator model:

$$\text{Max } \alpha^{\#} = \frac{1}{(n+1)(m+s)} \sum_{k=1}^{(n+1)(m+s)} \alpha_k$$
subject to:

$$\alpha_{ic} \leq \mu_{x_i^c} \leq 1, \quad i = 1, \ldots, m; \quad c = 1, \ldots, n$$
$$\alpha_{rc} \leq \mu_{y_r^c} \leq 1, \quad r = 1, \ldots, s; \quad c = 1, \ldots, n$$
$$\alpha_i \leq \mu_{x_i} \leq 1, \quad i = 1, \ldots, m$$
$$\alpha_r \leq \mu_{y_r} \leq 1, \quad r = 1, \ldots, s$$
$$\alpha_k \in [0, 1], \quad k = 1, \ldots,$$

$$\sum_{j=1}^{n}\lambda_j^c = 1, \quad c = 1, \ldots, n$$

$$\lambda_j^c \geq 0, \quad j = 1, \ldots, n; \quad c = 1, \ldots, n \qquad (15)$$

It is easy to understand that the optimal value $\alpha^{\#*}$ represents the total amount of all membership functions.

The average operator model (15) can be modified into following model:

$$\text{Max } \alpha^{\#} = \frac{1}{(n+1)(m+s)}\left(\sum_{c=1}^{n}\sum_{i=1}^{m}\mu_{x_i^c} + \sum_{c=1}^{n}\sum_{r=1}^{s}\mu_{y_r^c} + \sum_{i=1}^{m}\mu_{x_i} + \sum_{r=1}^{s}\mu_{y_r}\right)$$

subject to:

$$0 \leq \mu_{x_i^c} \leq 1, \quad i = 1, \ldots, m; \quad c = 1, \ldots, n$$

$$0 \leq \mu_{y_r^c} \leq 1, \quad r = 1, \ldots, s; \quad c = 1, \ldots, n$$

$$0 \leq \mu_{x_i} \leq 1, \quad i = 1, \ldots, m$$

$$0 \leq \mu_{y_r} \leq 1, \quad r = 1, \ldots, s$$

$$\sum_{j=1}^{n}\lambda_j^c = 1, \quad c = 1, \ldots, n$$

$$\lambda_j^c \geq 0, \quad j = 1, \ldots, n; \quad c = 1, \ldots, n \qquad (16)$$

The optimal value $\alpha^{\#*}$ generated by the model (16) stands for maximizing the total amount of membership functions, but is considered as fully compensatory model. In order to offer any desirable compromise solution between non-compensatory and fully compensatory to the decision maker can be developed a model with a parameter α' provided by decision maker. The new model is defined by (17):

$$\text{Max } \alpha^{\#} = \frac{1}{(n+1)(m+s)}\left(\sum_{c=1}^{n}\sum_{i=1}^{m}\mu_{x_i^c} + \sum_{c=1}^{n}\sum_{r=1}^{s}\mu_{y_r^c} + \sum_{i=1}^{m}\mu_{x_i} + \sum_{r=1}^{s}\mu_{y_r}\right)$$

subject to:

$$\alpha' \leq \mu_{x_i^c} \leq 1, \quad i = 1, \ldots, m; \quad c = 1, \ldots, n$$

$$\alpha' \leq \mu_{y_r^c} \leq 1, \quad r = 1, \ldots, s; \quad c = 1, \ldots, n$$

$$\alpha' \leq \mu_{x_i} \leq 1, \quad i = 1, \ldots, m$$

$$\alpha' \leq \mu_{y_r} \leq 1, \quad r = 1, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j^c = 1, \quad c = 1, \ldots, n$$

$$\lambda_j^c \geq 0, \quad j = 1, \ldots, n; \quad c = 1, \ldots, n \qquad (17)$$

In (Wu and Guu, 2001) it is proved that the model (17) guarantees the obtaining of a fuzzy-efficient solution.

Conclusion

The fuzzy dimension of the DEA models introduces subjectivity in the choice of the membership function, the bound on the inputs and outputs, the choice of the global targets, and the bound of the global targets. The activity levels λ_j^c require additional analysis following the evaluation of the membership functions. In the absence of an efficiency score the activity levels λ_j^c for each DMU reveal whether it is efficient or inefficient. For a DMU to be 100% efficient the activity level associated with it in the composite unit must attain the value one. This implies that such a DMU is its own "reference set" as it is 100% efficient relative to all the decision making units.

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