Income and Wealth Distribution in a Neoclassical Two-Sector Heterogeneous-Households Growth Model with Elastic Labor Supply and Consumer Durable Goods

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ABSTRACT

This paper proposes a two-sector two-group growth model with elastic labor supply and consumer durable goods. We study dynamics of wealth and income distribution in a competitive economy with capital accumulation as the main engine of economic growth. The model is built on the Uzawa two-sector model. It is also influenced by the neoclassical growth theory and the post-Keynesian theory of growth and distribution. We plot the motion of the economic system and determine the economic equilibrium. We carry out comparative dynamic analysis with regard to the propensity to save and improvements in human capital and technology.

KEYWORDS: *economic structure; heterogeneous households; endogenous time; durable consumer good; economic growth.*

JEL CLASSIFICATION: *O41; R22*

1. INTRODUCTION

The purpose of this study is to examine dynamic interdependence between economic growth and economic structural change with endogenous time distribution and durable consumer good. There are a few studies in the literature of theoretical economics. The study of the economic growth with income and wealth distribution has caused a lot of attention in economic growth theory. Yet, we argue that economics still needs an analytical framework for properly dealing with issues related to income and wealth distribution and economic growth with microeconomic foundation. It is well known that the traditional neoclassical growth theory has not adequately modeled economic structural change with heterogeneous capital on the basis of microeconomic foundation (Burmeister & Dobell, 1970; Jones & Manuelli, 1997). We build a two-sector two-group growth model to examine dynamics of wealth and income distribution with capital accumulation as the main engine of economic growth. Our approach is influenced by the neoclassical growth theory as well as by the post-Keynesian theory of growth and distribution (e.g., Pasinetti, 1974; Salvadori, 1991). In the most of the Post-Keynesian growth models with heterogeneous classes the economic system has a single production sector. There are exceptions, for instance, Stiglitz (1967) proposes a growth model of two sectors and two classes. The Stiglitz model synthesizes the post-Keynesian theory and Uzawa's two-sector model. But there are few further studies along the research line. This study deals with similar economic issues to those addressed by the Stiglitz model but in an alternative approach to household behavior. Moreover, labor supply is an endogenous variable in our model, while labor supply is fixed in the Stiglitz model.

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The core model in the neoclassical growth theory was proposed by Solow (1956). Since then the Solow model has been extended and generalized in different directions with regard to various economic issues. As the Solow model is proposed for one sector economy, it is not proper to deal with economic structural change and price changes of various goods. The initial extensions of the Solow model were proposed by Uzawa (1961) and Uzawa (1963), Meade (1961) and Kurz (1963). The Uzawa model has been applied as a standard model for modeling two-sector economic growth. The two sectors use capital and labor to produce, respectively, capital and consumer goods. In the traditional two-sector economy, output of the capital sector is used entirely for investment and that of the consumption sector for consumption. Economists have made many efforts in generalizing and extending the Uzawa two-sector model by, for instance, introducing more general production functions, money, externalities, knowledge, human capital, and fictions in different markets (for instance, Takayama, 1985; Galor, 1992; Azariadis, 1993; Mino, 1996; Drugeon & Venditti, 2001; Harrison, 2003; Cremers, 2006; Herrendorf & Valentinyi, 2006; Li & Lin, 2008; Stockman (2009). The two-sector economic structure is recognized as not proper for dealing issues of heterogeneous capital goods and multiple services. Many empirical studies demonstrate that economic sectors are different in their dynamic interactions with the economic system. Acconcia and Simonelli (2008: 3010) found out that the early literature on empirical studies of business fluctuations implicitly assumes that “a one-sector model is sufficient in order to correctly interpret the business cycle.” Nevertheless, the recent literature emphasizes the necessity of dividing production side into different sectors (Baxter, 1996; Whelan, 2003; Erceg et al. 2005; Fisher, 2006). This study introduces consumer durable goods into the traditional two-sector model. Consumer durable goods, such as cars, home appliances, consumer electronics, furniture, sports equipment, and toys, are important in terms of households’ expenditure shares in modern economies. The Uzawa model is not proper for dealing with non-durable and durable goods differently. This study applies Zhang’s approach to household behavior to extend the two-sector economic structure by distinguishing consumer non-durable goods and durable goods (Zhang, 1993; Zhang, 2005). We can also include housing as consume durable goods. Housing is essential for proper living. Housing is the largest component of nonhuman wealth for households. Housing is the largest expense for many households. For instance, as described in DiPasquale and Wheaton (1996) real estate investment is over 50 % of total private investment and real estate assets represent just under 60% of the nation’s wealth and almost 70% of U.S. real estate is residential. More than 60 percent of total net worth held by all households in Korea are housing assets (Cho, 2010). There are many studies on interactions between housing markets and macroeconomy (Henderson et al., 2001; Leung, 2004; Leamer, 2007; Ghent & Owyang, 2010). This study also provides some insights into demand and supply of housing markets. This study synthesizes the ideas in the two-sector model with endogenous labor by Zhang (2005), the growth model with durable consumer good by Zhang (2008), and the growth model with heterogeneous households by Zhang (2014). The remainder of this study is organized as follows. Section 2 defines the two-sector two-group model. Section 3 examines dynamic properties of the model. Section 4 carries out comparative dynamic analysis with regard to some parameters. The appendix gives the procedure for determining the two monotonous differential equations in section 3.

2. THE MODEL

The production side of the economic system is according to the Uzawa model (Burmesiter & Dobell, 1970; Barro & Sala-i-Martin, 1995). The two sectors produce respectively consumption and capital goods. We extend the Uzawa model in that capital goods can be used not only as input in the two sectors but also as consumer durable goods by the household. Capital is assumed to depreciate at a constant exponential rate δ_k.

2.1. Production sectors

The two sectors use labor and capital good as inputs. The two inputs are smoothly substitutable for each other in each sector and are freely transferable from one sector to the other. The technology of each sector is neoclassical. We use subscripts i and s to stand for respectively the capital goods sector and the consumer goods sector. The production functions are

$$F_q(t) = A_q K_q^{\alpha_q}(t) N_q^{\beta_q}(t), \quad q = i, s,$$

where F_q are the output of sector q, and $K_q(t)$ and $N_q(t)$ are respectively the capital and labor used by sector q.

We class the labor force into two groups. Let $T_j(t)$ stand for the work time of a representative household of group j and $N(t)$ for the flow of labor services used at time t for production. We assume that labor is always fully employed. We measure $N(t)$ as follows

$$N(t) = h_1 T_1(t) \bar{N}_1 + h_2 T_2(t) \bar{N}_2, \tag{1}$$

where h_j are the levels of human capital of group j, $j = 1, 2$. Markets are perfectly competitive. The capital good is selected serves as a medium of exchange and is taken as numeraire. Let $p(t)$ stand for the price of consumer good. The rate of interest $r(t)$ and wage rate $w(t)$ of per unit qualified labor force are determined by markets. Hence, for any individual firm $r(t)$ and $w(t)$ are given at each point of time. The production sector chooses the two variables $K_j(t)$ and $N_j(t)$ to maximize its profit. The marginal conditions are given by

$$r + \delta_k = f_i'(k_i) = p\, f_s'(k_s), \quad w(t) = f_i(k_i) - k_i f_i'(k_i) = p(t)\left[f_s(k_s) - k_s f_s'(k_s)\right], \tag{2}$$

where $f_q(t) \equiv F_q(t) / N_q(t)$. The wage rate of group j is equal to $w_j(t) = h_j w(t)$.

2.2. Full employment of the input factors

Total capital stock $K(t)$ is distributed between the two sectors and the households. Full employment of labor and capital is expressed as

$$K_i(t) + K_s(t) + \tilde{K}_1(t) + \tilde{K}_2(t) = K(t), \tag{3}$$

where $\tilde{K}_j(t)$ is the level of consumer durables used by group j, $j = 1, 2$. We rewrite (3) as

$$n_i(t) k_i(t) + n_s(t) k_s(t) + \tilde{n}_1(t) \tilde{k}_1(t) + \tilde{n}_2(t) \tilde{k}_2(t) = k(t), \quad n_i(t) + n_s(t) = 1, \tag{4}$$

where

$$k(t) \equiv \frac{K(t)}{N(t)}, \quad n_q(t) \equiv \frac{N_q(t)}{N(t)}, \quad q = i, s, \quad \tilde{k}_j(t) \equiv \frac{\tilde{K}_j(t)}{\bar{N}_j}, \quad \tilde{n}_j(t) \equiv \frac{\bar{N}_j}{N(t)}, \quad j = 1, 2.$$

We use $\bar{k}_j(t)$ to represent per capita wealth of group j at t. According to the definitions, we have

$$K(t) = \bar{k}_1(t)\bar{N}_1 + \bar{k}_2(t)\bar{N}_2. \qquad (5)$$

2.3. Current and disposable incomes

Consumers make decisions on choice of levels of consumer durables, services, commodities, and saving. This study uses the approach to consumers' behavior proposed by Zhang (1993) and Zhang (2005). Group j's per capita current income $y_j(t)$ from the interest payment $r(t)\bar{k}_j(t)$ and the wage payment $T_j(t)w_j(t)$ are

$$y_j(t) = r(t)\bar{k}_j(t) + T_j(t)w_j(t).$$

The sum that consumers spend for consuming and saving is not necessarily equal to the current income. Consumers may sell wealth to buy, for instance, goods for having a party at home. A retired man may live on the accumulated wealth in the past. What consumer j can sell is equal to $\bar{k}_j(t)$, if we don't allow borrowing from others. The per capita disposable income of consumer j is equal to the sum of the current income and the wealth, i.e.

$$\hat{y}_j(t) = y_j(t) + \bar{k}_j(t) = (1 + r(t))\bar{k}_j(t) + T_j(t)w_j(t), \quad j = 1, .2. \qquad (6)$$

The disposable income is spent on saving and consuming. The disposable income is distributed between saving $s_j(t)$, consumer durables $\tilde{k}_j(t)$, and consumer good $c_j(t)$. The budget constraint of consumer j is

$$p(t)c_j(t) + (r(t) + \delta_k)\tilde{k}_j(t) + s_j(t) = \hat{y}_j(t) = (1 + r(t))\bar{k}_j(t) + T_j(t)w_j(t). \qquad (7)$$

We use $T_{hj}(t)$ to represent the leisure time at time t and T_0 the (fixed) available time for work and leisure. The time constraint is

$$T_j(t) + T_{hj}(t) = T_0.$$

Substituting this relation into (7) yields

$$T_{hj}(t)w_j(t) + p(t)c_j(t) + (r(t) + \delta_k)\tilde{k}_j(t) + s_j(t) = \bar{y}_j(t) \equiv (1 + r(t))\bar{k}_j(t) + T_0 w_j(t). \qquad (8)$$

2.4. Utility function and optimal decision

The utility level $U_j(t)$ dependent on $c_j(t)$, $\tilde{k}_j(t)$ and $s_j(t)$. The utility level of group j is specified as follows:

$$U_j(t) = T_{hj}^{\sigma_{0j}}(t)c_j^{\xi_{0j}}(t)\tilde{k}_j^{\eta_{0j}}(t)s_j^{\lambda_{0j}}(t), \quad \sigma_{0j}, \xi_{0j}, \eta_{0j}, \lambda_{0j} > 0, \quad j = 1, ... J, \qquad (9)$$

where σ_{0j}, ξ_{0j}, η_{0j} and λ_{0j} are respectively group j's propensities to use leisure time, to consume consumer good, to utilize consumer durables, and to own wealth. Maximizing U_j subject to the budget constraints (8), we obtain the following first-order condition

$$w_j(t)T_{hj}(t) = \sigma_j \bar{y}_j(t), \quad p(t)c_j(t) = \xi_j \bar{y}_j(t), \quad (r(t) + \delta_k)\tilde{k}_j(t) = \eta_j \bar{y}_j(t), \quad s_j(t) = \lambda_j \bar{y}_j(t), \tag{10}$$

where

$$\sigma_j \equiv \rho_j \sigma_{0j}, \quad \xi_j \equiv \rho_j \xi_{0j}, \quad \mu_j \equiv \rho_j \mu_{0j}, \quad \lambda_j \equiv \rho_j \lambda_{0j}, \quad \rho_j \equiv \frac{1}{\sigma_{0j} + \xi_{0j} + \mu_{0j} + \lambda_{0j}}.$$

2.5. Wealth accumulation

According to the definitions of $s_j(t)$, the wealth accumulation of the consumer of group j is

$$\dot{\bar{k}}_j(t) = s_j(t) - \bar{k}_j(t). \tag{11}$$

Equation (11) implies that the change in wealth is equal the saving minus dissaving.

2.6. Demand and supply

What the consumer goods sector produces is consumed by the households

$$\sum_{j=1}^{J} c_j(t)\bar{N}_j = F_s(t). \tag{12}$$

The output of the capital goods sector is equal to the depreciation of capital stock and the net saving

$$\sum_{j=1}^{J} s_j(t)\bar{N}_j - K(t) + \delta_k K(t) = F_i(t). \tag{13}$$

We built the model economic growth model of physical capital distribution wealth and income distribution in the economy in which the markets are perfectly competitive. The model synthesizes some main ideas in economic growth theory. For instance, if we fix time distribution, assume a homogenous population, and neglect durable consumer good, our model is structurally similar to the neoclassical growth model by Solow (1956) and Uzawa (1961, 1963). If we fix time distribution and neglect durable consumer good, it is similar to the Stiglitz model (Stiglitz, 1967).

3. DYNAMICS AND EQUILIBRIUM OF THE ECONOMIC SYSTEM

This section examines dynamic properties of the two-sector two-group model. Although the system consists of many equations and interactions between variables are complicated, the following lemma shows that the motion of the economic system is solved with two differential equations.

Lemma

The motion of the two-sector two-group economic system is governed by the following two nonlinear differential equations with $k_i(t)$ and $k(t)$ as the variables

$$\dot{k}_i(t) = \Lambda(k_i(t),\, k(t)),$$
$$\dot{k}(t) = \Omega(k_i(t),\, k(t)), \qquad (14)$$

where Λ and Ω are functions of $k_i(t)$ and $k(t)$ defined in the appendix. Moreover, all the other variables can be determined as functions of $k_i(t)$ and $k(t)$ at any point in time by the following procedure: $\bar{k}_j(t),\ j=1,2$ by (A18) $\rightarrow$ $k_s(t)=\alpha k_i(t)$ $\rightarrow$ $f_q(t)$ by (A1) $\rightarrow$ $\bar{y}_j(t)$ by (A20) $\rightarrow$ $p(t)$ by (A3) $\rightarrow$ $n_i(t)$ and $n_s(t)$ by (A9) $\rightarrow$ $r(t)$ and $w(t)$ by (2) $\rightarrow$ $w_j(t)=h_j w(t),\ j=1,2$ $\rightarrow T_j(t)$ by (A12) $\rightarrow$ $\tilde{k}_j(t)$, $c_j(t)$ and $s_j(t)$ by (10) $\rightarrow$ $N(t)$ by (A16) $\rightarrow$ $\tilde{n}_j(t)=\bar{N}_j/N(t)$ $\rightarrow$ $N_q(t)=n_q(t)N_q(t),\ q=i,s$ $\rightarrow$ $K(t)=k(t)N(t)$ $\rightarrow$ $K_q(t)=k_q(t)N_q(t)$ $\rightarrow F_q(t)=f_q(t)N_q(t)$ $\rightarrow$ $U_j(t)$ by (9).

As the lemma provides the procedure to determine all the variables, it is straightforward to follow the motion of the system with computer. For simulation, we specify the parameter values

$$\bar{N}_1=6,\ \ \bar{N}_2=18,\ \ A_i=2,\ \ A_s=1,\ \ \alpha_i=0.38,\ \ \alpha_s=0.34,\ \ h_1=2,\ \ h_2=0.5,\ \ \delta_k=0.05,$$
$$\lambda_{01}=0.7,\ \ \sigma_{01}=0.1,\ \ \eta_{01}=0.1,\ \ \xi_{01}=0.1,\ \ \lambda_{02}=0.65,\ \ \sigma_{02}=0.1,\ \ \eta_{02}=0.07,\ \ \xi_{02}=0.08. \qquad (15)$$

Group $2's$ population is 3 times as many as group $1's$. Group $1's$ level of human capital is 4 times as high as group $2's$. Group 1 has lower (relative) propensity to save (which equals 0.7) than group $2's$ (which equals $0.65/0.9 \approx 0.72$) and group 1 has lower propensity to enjoy leisure time than group 2. We choose the initial conditions, $k(0)=18$ and $k_i(0)=8$. The motion of the variables is plotted in Figure 1 with Mathematica.

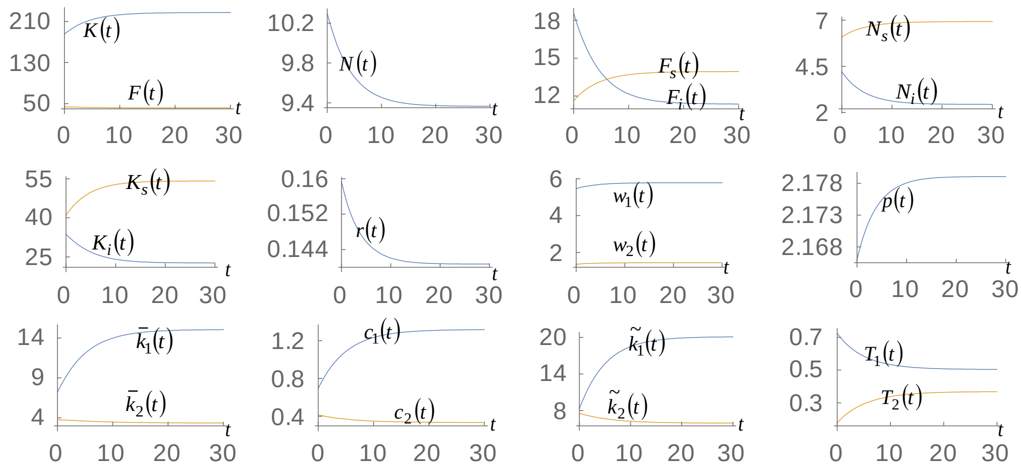

Figure 1. The motion of the economic system

Source: author

In Figure 1, $F(t) \equiv F_i(t) + p(t)F_s(t)$ stands for the national product. The labor share of the service sector rises as time passes. Group 1 works shorter hours, while Group 2 works longer hours. The price of the consumer good changes slightly over time. The wealth per capita, the consumption levels of durable goods and consumer goods and wage rate of group 1 are augmented, while the corresponding variables of group 2 are lessened. Hence, due to the initial conditions group 2 works longer hours, but their living conditions and wealth are not improved. The rate of interest lowers in associating in rising in the capital intensities. The output and two inputs of the consumer goods sector are increased, while the corresponding variables of the capital goods sector are lessened. The national output falls. As the per capita wealth of the economy is enhanced and the per capita wealth group 2 is reduced, the wealth capita is enlarged.

From Figure 1 we see that all the variables become stationary in the long term. This implies that the system has an equilibrium point. We now show that the system has a unique equilibrium point. By (11) we have $s_j = \bar{k}_j$ at equilibrium. From $s_j = \bar{k}_j$ and $s_j = \lambda_j \bar{y}_j$, we solve $\bar{k}_j = \lambda_j \bar{y}_j$. From $\bar{k}_j = \lambda_j \bar{y}_j$ and the definition of $\bar{y}_j$, we have

$$\bar{k}_j = \phi_j(k_i) \equiv \frac{\bar{h}_j k_i}{\bar{\lambda}_j k_i^{\beta_i} - \alpha_i A_i}, \qquad (16)$$

where $\bar{\lambda}_j \equiv 1/\lambda_j - 1 + \delta_k$ and $\bar{h}_j \equiv T_0 \beta_i A_i h_j$. From the definition of k and (5), we have

$$k = \left(\sum_{j=1}^{J} \bar{k}_j \bar{N}_j \right) \frac{1}{N}.$$

By this equation, we can rewrite (12) and (13) as

$$n_s = \frac{\varphi_s(k_i)}{N}, \quad n_i = \frac{\varphi_i(k_i)}{N}, \qquad (17)$$

where we use $c_j = \xi_j \bar{k}_j / \lambda_j p$, $s_j = \bar{k}_j$ and

$$\varphi_s(k_i) \equiv \left(\sum_{j=1}^{J} \frac{\bar{N}_j \xi_j \bar{k}_j}{\lambda_j} \right) \frac{1}{p\, f_s}, \quad \varphi_i(k_i) \equiv \left(\sum_{j=1}^{J} \bar{N}_j \bar{k}_j \right) \frac{\delta_k}{f_i}.$$

Substituting $\bar{k}_j = \lambda_j \bar{y}_j$ and $r + \delta_k = f_i'(k_i)$ into $(r + \delta_k)\tilde{k}_j = \eta_j \bar{y}_j$, we solve

$$\tilde{k}_j = \frac{\eta_j \bar{k}_j}{\lambda_j\, f_i'(k_i)}. \qquad (18)$$

From (17) and $n_i + n_s = 1$, we have

$$N = \varphi(k_i) \equiv \varphi_s(k_i) + \varphi_i(k_i). \qquad (19)$$

By $w_j T_{hj} = \sigma_j \bar{y}_j$, $\bar{k}_j = \lambda_j \bar{y}_j$ and $w_j = h_j \beta_i f_i$, we solve

$$T_{hj} = \frac{\bar{\sigma}_j \bar{k}_j}{f_i}, \tag{20}$$

where $\bar{\sigma}_j \equiv \sigma_j / \lambda_j h_j \beta_i$. From (20) and the definition of N, we solve

$$N = \sum_{j=1}^{J} \left(T_0 - \frac{\bar{\sigma}_j \bar{k}_j}{f_i} \right) h_j \bar{N}_j, \tag{21}$$

where we also use $T_j(t) + T_{hj}(t) = T_0$. Equalizing the right-hand sides of (19) and (21) yields

$$\Omega(k_i) \equiv \varphi(k_i) - \sum_{j=1}^{J} \left(T_0 - \frac{\bar{\sigma}_j \bar{k}_j}{f_i} \right) h_j \bar{N}_j = 0. \tag{22}$$

Summarizing the above analysis, we can determine all the variables at the equilibrium point by the following computing procedure: k_i by (22) $\rightarrow$ $\bar{k}_j$ by (16) $\rightarrow$ $k_s = \alpha k_i$ $\rightarrow$ f_j by (A1) $\rightarrow$ $\bar{y}_j = \bar{k}_j / \lambda_j$ $\rightarrow$ p by (A3) $\rightarrow$ N by (19) $\rightarrow$ n_j by (17) $\rightarrow$ $N_j = n_j N$ $\rightarrow$ r and w by (2) $\rightarrow$ $w_j = h_j w$ $\rightarrow T_{hj}$ by (20) $\rightarrow$ $\tilde{k}_j$, c_j and s_j by (10) $\rightarrow$ $K = kN$ $\rightarrow$ $K_j = k_j N_j$ $\rightarrow F_j = f_j N_j$.

Figure 2 demonstrates that equation (22) has a unique meaningful solution. The equation has actually two positive solutions, $k_i = 2.29$, $k_i = 9.30$, as demonstrated in Figure 1. Nevertheless, at $k_i = 2.29$ the work times of the two groups do not belong to $[0, 1]$.

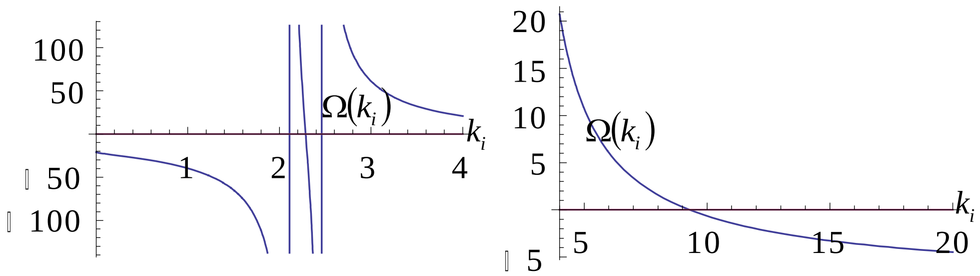

Figure 2. A unique meaningful solution of equation (22)
Source: author

Under (15), we have the equilibrium values in (23). As $p = 2.18$ and $F_s = 13.9$, we see that the consumer goods sector makes more contribution to the national economy than the capital goods sector ($F_i = 11.37$). Most of the labor force is employed by the consumer goods sector. The representative household from group 1 consumes more consumer good, holds more wealth, utilizes more consumer durables than the household from group 2. The representative household of group 2 works less hours than group 1.

$$F = 41.74, \;\; F_i = 11.37, \;\; F_s = 13.94, \;\; K = 227.40, \;\; K_i = 22.65, \;\; K_s = 54.14,$$

$$N = 9.37,\ N_i = 2.44,\ N_s = 6.93,\ f_i = 4.67,\ f_s = 2.01,\ k_i = 9.30,\ k_s = 7.81,$$
$$n_i = 0.26,\ n_s = 0.74,\ p = 2.18,\ r = 0.141,\ w_1 = 5.79,\ w_2 = 1.45,\ c_1 = 1.32,\ c_2 = 0.34,$$
$$T_{h1} = 0.50,\ T_{h2} = 0.63,\ \bar{k}_1 = 20.10,\ \bar{k}_2 = 5.93,\ \tilde{k}_1 = 15.06,\ \tilde{k}_2 = 3.35. \quad (23)$$

The two eigenvalues are $\{-0.293,\ 3\times10^{-17}\}$.

4. COMPARATIVE DYNAMIC ANALYSIS

We plotted the transitional paths of the economic system with some initial conditions and determine the equilibrium values of the variables. We now examine how a change in a parameter will shift the transitional paths and affect the equilibrium values of the variables.

4.1. An improvement in the capital goods sector's productivity

We first study what will happen to the economic system if we allow the capital goods sector's total productivity is enhanced in the following way: $A_i: 2 \Rightarrow 2.1$. Following the lemma, we plot the shifts of the transitional paths as in Figure 3. As the productivity is improved, the capital goods sector's output level is enhanced. Although the total capital is increased, the rate of interest rises initially but subsequently comes back to its original value. In the long term the change in the total productivity has no impact on the rate of interest. As the capital supply is increased, the price of consumer goods is enhanced. Although the two groups' wage rates are changed in the same rate (because the change is due to the total productivity improvement in the Cobb-Douglas production function), group $1's$ wage income is increased far more than group $2's$, irrespective of that group $2's$ works longer hours than before. Group $1's$ wage rate, consumption levels of consumer good and durable goods, and per capita wealth are all increased, but the values of the corresponding variables are lowered initially but subsequently augmented. The total labor supply and two sector's labor inputs are all increased initially but end in the original values in the long term. This occurs because in the long term each household's time distribution is invariant in the long term. The national output and the output of the capital goods sector rise, but the output level of the consumer goods sector fall initially and subsequently rise. This occurs in association with the shift of the labor distribution. Initially the share of labor force is distributed more to the capital goods sector but subsequently is little affected.

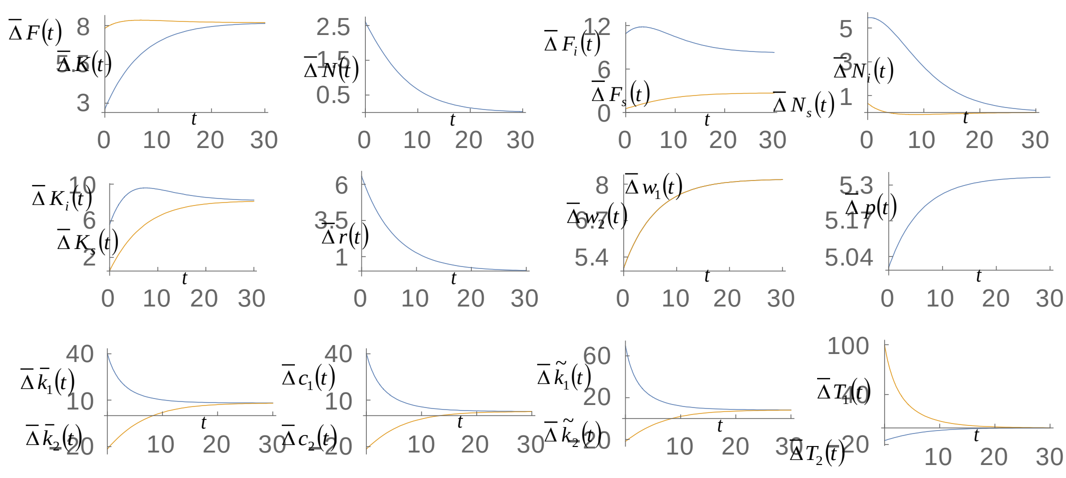

Figure 3. An Improvement in the capital goods sector's technology
Source: author

4.2. An enhancement in group $1's$ human capital

We now deal with the case that group $1's$ human capital is enhanced in the following way: $h_1 : 2 \Rightarrow 2.2$. The immediate result of a rise in group $1's$ human capital is the rise in the group's wage rate. This leads to a slight falling in the group $2's$ wage rate because the gap in human capital is enlarged. Group 1 works longer hours as the opportunity cost of leisure falls; while group 2 works shorter hours in association with falling in the group's opportunity cost of leisure. Nevertheless, the work hours are slightly affected by the parameter change. The price of consumer good is slightly affected. The human capital improvement enhances the total labor supply and labor inputs of the two sectors. The share of the labor force of the consumer goods sector is increased. The enhanced human capital reduces the capital intensities of the two sectors, which is associated with the rising in the rate of interest. From the group's improved human capital group 1 benefits in terms of wealth, consumption levels of the two goods in the transitional process as well as the long term. But this does not happen to the other group. Group 1 benefits in the transitional process but not in the long term.

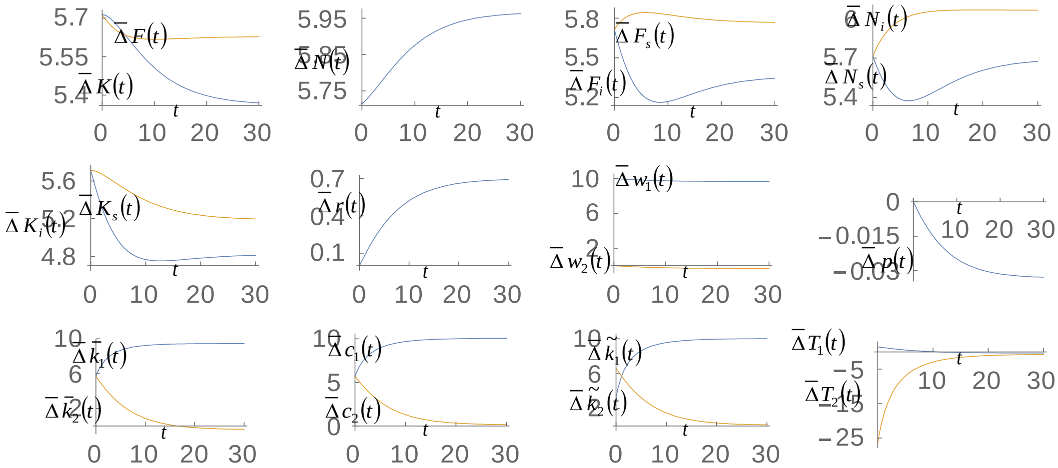

Figure 4. An enhancement in group 1's human capital

Source: author

4.3. An enhancement in group 2's propensity to save

We now study the effects of the following rise in group $2's$ propensity to save: $\lambda_{02} : 0.65 \Rightarrow 0.69$. As group 1 increases the propensity to save, the per capita wealth of the group falls initially but subsequently rises. This occurs as the increased propensity to save leads to the enhancement in the total capital in association of falling in the rate of interest. As more capital is saved, the price of consumer good is enhanced. The augmented capital and labor supply increase the output levels of the two sectors. We conclude that even group 2 does not benefit initially, in the long term all the households in the economy benefit.

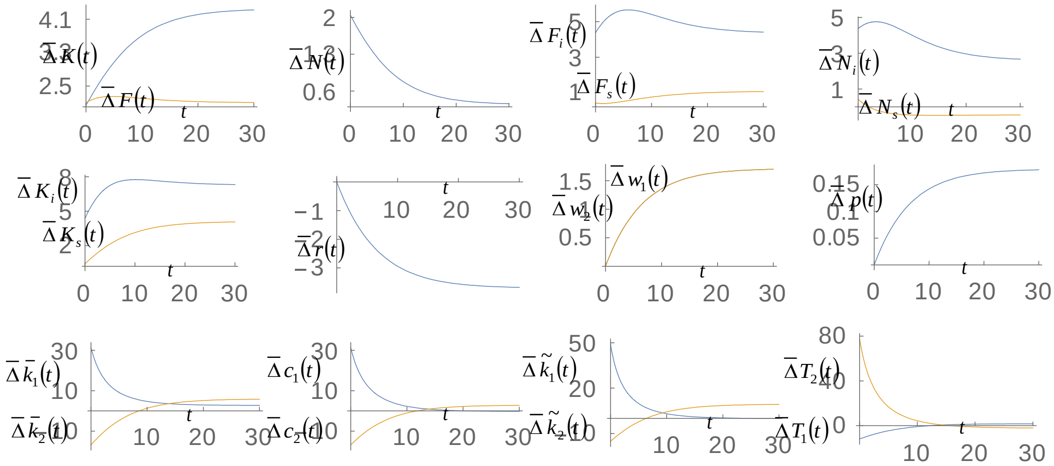

Figure 5. Group 2's propensity to save being enhanced
Source: author

5. CONCLUSIONS

This paper introduced heterogeneous households, endogenous time and durable consumer good into the traditional Uzawa two-sector growth model. We dealt with the complicated economic issues with an alternative utility function. This study does not only model the economic structure like the Uzawa's model, but also determines endogenous income and wealth distribution between heterogeneous households. The model is influenced by the neoclassical growth theory and the post-Keynesian theory of growth and distribution. We plotted the motion of the economic system and determined the economic equilibrium. We also carry out comparative dynamic analysis with regard to the propensity to save, improvement in human capital, and enhanced technology. As our model is influenced by the core models in the neoclassical and post-Keynesian approaches and these core models have been extended and generalized in different ways, it is not difficult to generalize our model on the basis of these extensions and generalizations. We may generalize the model by using more general forms of production or utility functions. Our classification of the production into two sectors and the population into two groups should be generalized. For instance, we may introduce externalities into the two sector model on the basis of the approaches taken by, for instance, Benhabib and Farmer (1994) and Boldrin et al. (2001).

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REFERENCES

Acconcia, A. & Simonelli, S. (2008). Interpreting aggregate fluctuations looking at sectors. *Journal of Economic Dynamics & Control*, *32*(9), 3009-31.
Azariadis, C. (1993). *Intertemporal macroeconomics*. Oxford: Blackwell.
Barro, R.J. & Sala-i-Martin, X. (1995). *Economic growth*. New York: McGraw-Hill, Inc.
Baxter, M. (1996). Are consumer durables important for business cycles? *The Review of Economics and Statistics*, *78*(1), 147-55.

Benhabib, J. & Farmer, R.E.A. (1994). Indeterminacy and increasing returns. *Journal of Economic Theory*, *63*(1), 19-41.

Boldrin, M., Nishimura, K., Shigoka, T., & Yano, M. (2001). Chaotic equilibrium dynamics in endogenous growth models. *Journal of Economic Theory, 96*(1-2), 97-132.

Burmeister, E. & Dobell, A.R. (1970). *Mathematical theories of economic growth*. London: Collier Macmillan Publishers.

Cho, S.W. (2010). Household wealth accumulation and portfolio choices in Korea. *Journal of Housing Economics*, *19*(1), 13-25.

Cremers, E.T. (2006). Dynamic efficiency in the two-sector overlapping generalizations model. *Journal of Economic Dynamics & Control*, *30*(10), 1915-36.

DiPasquale, D. & Wheaton, W.C. (1996). *Urban economics and real estate markets*. Englewood Cliffs: Prentice Hall.

Drugeon, J.P. & Venditti, A. (2001). Intersectoral external effects, multiplicities & indeterminacies. *Journal of Economic Dynamics & Control*, *25*(5), 765-787.

Erceg, C., Guerrieri, L., & Gust, C. (2005). Can long-run restrictions identify technology shocks? *Journal of European Economic Association*, *3*(6), 1237-78.

Harrison, S.G. (2003). Returns to scale and externalities in the consumption and investment sectors. *Review of Economic Dynamics*, *6*(4), 963-76.

Herrendorf, B. & Valentinyi, A. (2006). On the stability of the two-sector neoclassical growth model with externalities. *Journal of Economic Dynamics and Control, 30*(8), 1339-61.

Fisher, J.D.M. (2006). The dynamic effects of neutral and investment-specific technology shocks. *Journal of Political Economy*, *114*(3), 413-51.

Galor, O. (1992). Two-sector overlapping-generations model: A global characterization of the dynamical system. *Econometrica, 60*(6), 1351-86.

Ghent, A.C. & Owyang, M.T. (2010). Is housing the business cycle? Evidence from U.S. cities. *Journal of Urban Economics, 67*(3), 336-51.

Henderson, J.V., Schalizi, Z., & Venables, A.J. (2001). Geography and development. *Journal of Economic Geography*, *1*(1), 81-105.

Jones, L. & Manuelli, R.E. (1997). The sources of growth. *Journal of Economic Dynamics and Control*, 21(1), 75-114.

Kurz, M. (1963). The two sector extension of Swan's model of economic growth: The case of no technical change. *International Economic Review*, 4(1), 1-12.

Leamer, E.E. (2007). Housing is the business cycle. *NBER Working Paper*, No. 13428.

Leung, C. (2004). Macroeconomics and housing: A review of the literature. *Journal of Housing Economics*, *13*(4), 249-67.

Li, J.L. & Lin S.L. (2008). Existence and uniqueness of steady-state equilibrium in a two-sector overlapping generations model. *Journal of Economic Theory*, *141*(1), 255-75.

Meade, J.E. (1961). *A neoclassical theory of economic growth*. London: Allen and Unwin.

Mino, K. (1996). Analysis of a two-sector model of endogenous growth with capital income taxation. *International Economic Review*, *37*(1), 227-51.

Pasinetti, L. (1962). Rate of profit and income distribution in relation to the rate of economic growth. *Review of Economic Studies, 29*(4), 267-79.

Salvadori, N. (1991). Post-Keynesian theory of distribution in the long run. In E.J. Nell and W. Semmler (Eds.), *Nicholas Kaldor and mainstream economics - Confrontation or convergence*? London: Macmillan.

Solow, R. (1956). A contribution to the theory of growth. *Quarterly Journal of Economics*, *70*(1), 65-94.

Stiglitz, J.E. (1967). A two sector two class model of economic growth. *Review of Economic Studies*, *34*(2), 227-38.

Stockman, D.R. (2009). Chaos and sector-specific externalities. *Journal of Economic Dynamics and Control*, *33*(12), 2030-46.
Takayama, A. (1985). *Mathematical economics*. Cambridge: Cambridge University Press.
Uzawa, H. (1961). On a two-sector model of economic growth I. *Review of Economic Studies*, *29*(1), 47-70.
Uzawa, H. (1963). On a two-sector model of economic growth II. *Review of Economic Studies*, *30*(2), 105-18.
Whelan, K. (2003). A two-sector approach to modeling U.S. NIPA data. *Journal of Money, Credit and Banking*, *35*(4), 627-56.
Zhang, W.B. (1993). Woman's labor participation and economic growth - Creativity, knowledge utilization and family preference. *Economics Letters*, *42*(1), 105-10.
Zhang, W.B. (2005). A two-sector growth model with labour supply. *Australian Journal of Labor Economics*, *8*(3), 245-60.
Zhang, W.B. (2008). Growth and residential distribution with economic structure and amenity: A synthesis of Solow-Uzawa's growth, Alonso's urban, and Muth's housing models. *Papers in Regional Science*, *87*(November), 277-303.
Zhang, W.B. (2014). Land value and rent dynamics in an integrated Walrasian general equilibrium and neoclassical growth theory. *Annals of the Alexandru Ioan Cuza University of Iasi*, *Economic Sciences Section*, *61*(2), 235-58.

APPENDIX

The production functions are expressed as

$$f_j(k_j) = A_j k_j^{\alpha_j}, \;\; A_j > 0, \;\; 0 < \alpha_j < 1, \;\; j = i, s. \qquad \text{(A1)}$$

From the equations (A1) and (2), we directly obtain

$$k_s = \alpha k_i, \qquad \text{(A2)}$$

where

$$\alpha \equiv \frac{\beta_i \alpha_s}{\beta_s \alpha_i}, \;\; \beta_j \equiv 1 - \alpha_j, \;\; j = i, s.$$

The capital intensity of the consumer goods sector is proportional to that of the capital goods sector. By $k_s = \alpha k_i$ and $\beta_i f_i = \beta_s p f_s$, we solve

$$w = \beta_i f_i, \;\; p = \frac{\beta_i A_i}{\beta_s \alpha^{\alpha_s} A_s} k_i^{\alpha_i - \alpha_s}. \qquad \text{(A3)}$$

The price of consumer good is positively related to the technological level of the capital goods sector but negatively related to that of the consumer goods sector. The price is positively or negatively related to the capital intensity of the capital goods sector, depending on the sign of $\alpha_i - \alpha_s$. If $\alpha_i = \alpha_s$, then the price is constant, $p = A_i / A_s$. In the remainder of this section, we require $\alpha_i > \alpha_s$. If the equality holds, then labor distribution is invariant in time. The analysis of $\alpha_i \le \alpha_s$ is similar. The condition $\alpha_i > \alpha_s$ (which implies $\alpha < 1$) guarantees $k_s(t) < k_i(t)$ for any t.

According to the definitions, we can rewrite (12) and (13) as

$$\sum_{j=1}^{2}\tilde{n}_j(t)c_j(t) = n_s(t)f_s(t),$$

$$\sum_{j=1}^{2}\tilde{n}_j(t)s_j(t) - \delta k(t) = n_i(t)f_i(t). \qquad \text{(A4)}$$

Substitute $pc_j = \xi_j \bar{y}_j$ and $s_j = \lambda_j \bar{y}_j$ into (A4)

$$\begin{bmatrix} \xi_1 \tilde{n}_1 & \xi_2 \tilde{n}_2 \\ \lambda_1 \tilde{n}_1 & \lambda_2 \tilde{n}_2 \end{bmatrix} \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = \begin{bmatrix} pn_s f_s \\ n_i f_i + \delta k \end{bmatrix}, \qquad \text{(A5)}$$

where we omit time in the expressions. We solve the above linear equations as

$$\bar{y}_1 = \frac{\beta_i \lambda_2 n_s f_i / \beta_s - (n_i f_i + \delta k)\xi_2}{\lambda \tilde{n}_1},$$

$$\bar{y}_2 = \frac{\xi_1 (n_i f_i + \delta k) - \beta_i \lambda_1 n_s f_i / \beta_s}{\lambda \tilde{n}_2}, \qquad \text{(A6)}$$

where we use $pf_s = \beta_i f_i / \beta_s$ and $\lambda \equiv \xi_1 \lambda_2 - \xi_2 \lambda_1 (\neq 0$ assumed). It should be noted that we did not discuss mathematical conditions under which it is meaningful to obtain some expressions. It is tedious to examine these conditions because of lengthy and complicated expressions. Substituting (A6) and $r + \delta_k = f_i^{'}(k_i)$ into $(r + \delta_k)\tilde{k}_j = \eta_j \bar{y}_j$, we solve

$$\tilde{k}_1 = (\bar{\xi}_2 n_s f_i - n_i f_i - \delta k)\frac{\xi_2 \tilde{\lambda}_1 k_i^{\beta_i}}{\tilde{n}_1},$$

$$\tilde{k}_2 = (n_i f_i + \delta k - \bar{\xi}_1 n_s f_i)\frac{\xi_1 \tilde{\lambda}_2 k_i^{\beta_i}}{\tilde{n}_2}, \qquad \text{(A7)}$$

where

$$\tilde{\lambda}_j \equiv \frac{\eta_j}{\lambda \alpha_i A_i}, \quad \bar{\xi}_j \equiv \frac{\lambda_j \beta_i}{\xi_j \beta_s}, \quad j = 1, 2.$$

Substitute (A7) into (4)

$$\psi_i(k_i)n_i + \psi_s(k_i)n_s = k, \qquad \text{(A8)}$$

where

$$\psi_i(k_i) \equiv \left(\frac{1 - A_i \xi_2 \tilde{\lambda}_1 + A_i \xi_1 \tilde{\lambda}_2}{1 + \delta \xi_2 \tilde{\lambda}_1 k_i^{\beta_i} - \delta \xi_1 \tilde{\lambda}_2 k_i^{\beta_i}}\right)k_i, \quad \psi_s(k_i) \equiv \left(\frac{\alpha + A_i \bar{\xi}_2 \xi_2 \tilde{\lambda}_1 - A_i \bar{\xi}_1 \xi_1 \tilde{\lambda}_2}{1 + \delta \xi_2 \tilde{\lambda}_1 k_i^{\beta_i} - \delta \xi_1 \tilde{\lambda}_2 k_i^{\beta_i}}\right)k_i.$$

From (A8) and $n_i + n_s = 1$, we get

$$n_i = \bar{\psi}_i(k_i, k) \equiv \frac{\psi_s(k_i) - k}{\psi_s(k_i) - \psi_i(k_i)}, \quad n_s = \bar{\psi}_s(k_i, k) \equiv \frac{k - \psi_i(k_i)}{\psi_s(k_i) - \psi_i(k_i)}. \qquad \text{(A9)}$$

The labor distribution between the two sectors is uniquely determined by k and k_i. We consider $n_i(t)$ and $n_s(t)$ as functions of $k(t)$ and $k_i(t)$.

Substitute (A6) into $w_j T_{hj} = \sigma_j \bar{y}_j$

$$w_1 T_{h1} = \frac{\beta_i \sigma_1 \lambda_2 n_s f_i - (n_i f_i + \delta k)\beta_s \sigma_1 \xi_2}{\beta_s \lambda \tilde{n}_1},$$

$$w_2 T_{h2} = \frac{(n_i f_i + \delta k)\beta_s \sigma_2 \xi_1 - \beta_i \sigma_2 \lambda_1 n_s f_i}{\beta_s \lambda \tilde{n}_2}, \tag{A10}$$

where we use $pf_s = \beta_i f_i / \beta_s$. Inserting $T_j + T_{hj} = T_0$, $w_j = h_j \beta_i A_i k_i^{\alpha_i}$ and

$$\tilde{n}_j = \frac{\bar{N}_j}{\sum_{j=1}^{2} h_j \bar{N}_j T_j},$$

in (A10), we obtain

$$T_0 = (\Gamma_1 h_1 \bar{N}_1 + 1)T_1 + \Gamma_1 h_2 \bar{N}_2 T_2,$$

$$T_0 = h_1 \bar{N}_1 \Gamma_2 T_1 + (h_2 \bar{N}_2 \Gamma_2 + 1)T_2, \tag{A11}$$

where we use $T_j + T_{hj} = T_0$ and

$$\Gamma_1(k_i, k) \equiv \frac{\beta_i \sigma_1 \lambda_2 n_s f_i - (n_i f_i + \delta k)\beta_s \sigma_1 \xi_2}{\beta_s \lambda\, h_1 \beta_i A_i \bar{N}_1 k_i^{\alpha_i}}, \quad \Gamma_2(k_i, k) \equiv \frac{(n_i f_i + \delta k)\beta_s \sigma_2 \xi_1 - \beta_i \sigma_2 \lambda_1 n_s f_i}{h_2 \beta_i A_i \beta_s \lambda \bar{N}_2 k_i^{\alpha_i}}.$$

Solving (A11) with T_j as variables, we have

$$T_1 = \Gamma_{T1}(k_i, k) \equiv \left(\frac{h_2 \bar{N}_2 \Gamma_2 - \Gamma_1 h_2 \bar{N}_2 + 1}{(h_2 \bar{N}_2 \Gamma_2 - \Gamma_1 h_2 \bar{N}_2 + 1)h_1 \bar{N}_1 \Gamma_2 + (h_2 \bar{N}_2 \Gamma_2 + 1)(\Gamma_1 h_1 \bar{N}_1 - h_1 \bar{N}_1 \Gamma_2 + 1)} \right) T_0,$$

$$T_2 = \Gamma_{T2}(k_i, k) \equiv \left(\frac{\Gamma_1 h_1 \bar{N}_1 - h_1 \bar{N}_1 \Gamma_2 + 1}{(h_2 \bar{N}_2 \Gamma_2 - \Gamma_1 h_2 \bar{N}_2 + 1)h_1 \bar{N}_1 \Gamma_2 + (h_2 \bar{N}_2 \Gamma_2 + 1)(\Gamma_1 h_1 \bar{N}_1 - h_1 \bar{N}_1 \Gamma_2 + 1)} \right) T_0.$$

(A12)

Insert $s_j = \lambda_j \bar{y}_j$ into the second equation in (13)

$$\sum_{j=1}^{2} \tilde{n}_j \lambda_j \bar{y}_j - \delta k = A_i k_i^{\alpha_i} n_i. \tag{A13}$$

Substituting (6) into (A13) yields

$$\sum_{j=1}^{2} \tilde{n}_j \lambda_j (1+r) \bar{k}_j + \sum_{j=1}^{2} \tilde{n}_j \lambda_j w_j - \delta k = A_i k_i^{\alpha_i} n_i .$$

Insert $r = \alpha_i A_i k_i^{-\beta_i} - \delta_k$ and $w_j = h_j \beta_i A_i k_i^{\alpha_i}$ in the above equation

$$\sum_{j=1}^{2} \tilde{n}_j \lambda_j \left(\alpha_i A_i k_i^{-\beta_i} + \delta \right) \bar{k}_j + \beta_i A_i k_i^{\alpha_i} \sum_{j=1}^{2} \tilde{n}_j \lambda_j h_j - \delta k = A_i k_i^{\alpha_i} n_i . \qquad \text{(A14)}$$

Insert n_i in (A8) in (A10)

$$\sum_{j=1}^{2} \tilde{n}_j \lambda_j \left(\alpha_i A_i k_i^{-\beta_i} + \delta \right) \bar{k}_j + \left(\beta_i A_i \sum_{j=1}^{2} \tilde{n}_j \lambda_j h_j - \frac{A_i \psi_s(k_i)}{\psi_s(k_i) - \psi_i(k_i)} \right) k_i^{\alpha_i} = \left(\delta - \frac{A_i k_i^{\alpha_i}(k_i)}{\psi_s(k_i) - \psi_i(k_i)} \right) k . \qquad \text{(A15)}$$

According to the definitions, we can rewrite (5) as

$$k \hat{N}(k_i, k) = \sum_{j=1}^{2} \bar{N}_j \bar{k}_j , \qquad \text{(A16)}$$

where

$$\hat{N}(k_i, k) \equiv h_1 \bar{N}_1 \Gamma_{T1}(k_i, k) + h_2 \bar{N}_2 \Gamma_{T2}(k_i, k).$$

Substitute (A16) into (A15)

$$\Omega_1(k_i) \bar{k}_1 + \Omega_2(k_i) \bar{k}_2 = \Omega_0(k_i, k), \qquad \text{(A17)}$$

where we use $\tilde{n}_j = \bar{N}_j / N$ and

$$\Omega_j(k_i) \equiv \left(\frac{A_i k_i^{\alpha_i}}{\psi_s(k_i) - \psi_i(k_i)} + \lambda_j \alpha_i A_i k_i^{-\beta_i} + \lambda_j \delta - \delta \right) \bar{N}_j , \quad j = 1, 2,$$

$$\Omega_0(k_i, k) \equiv \left(\frac{A_i \psi_s(k_i) \hat{N}(k_i, k)}{\psi_s(k_i) - \psi_i(k_i)} - \beta_i A_i \sum_{j=1}^{2} \lambda_j h_j \bar{N}_j \right) k_i^{\alpha_i} .$$

We solve (A16) and (A17) with $\bar{k}_1$ and $\bar{k}_2$ as variables

$$\bar{k}_1 = \bar{\Omega}_1(k_i, k) \equiv \frac{\bar{N}_2 \Omega_0(k_i, k) - k \Omega_2(k_i) \hat{N}(k_i, k)}{\bar{N}_2 \Omega_1(k_i) - \bar{N}_1 \Omega_2(k_i)},$$

$$\bar{k}_2 = \bar{\Omega}_2(k_i, k) \equiv \frac{k \hat{N}(k_i, k) \Omega_1(k_i) - \bar{N}_1 \Omega_0(k_i, k)}{\bar{N}_2 \Omega_1(k_i) - \bar{N}_1 \Omega_2(k_i)}. \qquad \text{(A18)}$$

Hence, we can express $\bar{k}_1(t)$ and $\bar{k}_2(t)$ as functions of $k_i(t)$ and $k(t)$ at any point of time. Taking derivatives of (A18) with respect to t yields.

$$\dot{\bar{k}}_1 = \overline{\Omega}_{1k_i}(k_i, k)\dot{k}_i + \overline{\Omega}_{1k}(k_i, k)\dot{k},$$

$$\dot{\bar{k}}_2 = \overline{\Omega}_{2k_i}(k_i, k)\dot{k}_i + \overline{\Omega}_{2k}(k_i, k)\dot{k}. \qquad \text{(A19)}$$

Insert (A9) and $\tilde{n}_j = \overline{N}_j / \hat{N}(k_i, k)$ into (A6)

$$\bar{y}_1 = \left(\frac{A_i\beta_i\lambda_2 \overline{\psi}_s(k_i)k_i^{\alpha_i} - \left(A_i\overline{\psi}_i(k_i)k_i^{\alpha_i} + \delta k\right)\beta_s\xi_2}{\lambda\beta_s\overline{N}_1} \right)\hat{N}(k_i, k),$$

$$\bar{y}_2 = \left(\frac{\left(A_i\overline{\psi}_i(k_i)k_i^{\alpha_i} + \delta k\right)\beta_s\xi_1 - A_i\beta_i\lambda_1\overline{\psi}_s(k_i)k_i^{\alpha_i}}{\lambda\beta_s\overline{N}_2} \right)\hat{N}(k_i, k). \qquad \text{(A20)}$$

We can express $\bar{y}_j(t)$ as functions of $k_i(t)$ and $k(t)$.

Insert $s_j = \lambda_j\bar{y}_j$ and (A20) into (11)

$$\dot{\bar{k}}_1 = \tilde{s}_1(k_i, k) \equiv \left(\frac{A_i\beta_i\lambda_2 \overline{\psi}_s(k_i)k_i^{\alpha_i} - \left(A_i\overline{\psi}_i(k_i)k_i^{\alpha_i} + \delta k\right)\beta_s\xi_2}{\lambda\beta_s\overline{N}_1} \right)\lambda_1\hat{N}(k_i, k) - \overline{\Omega}_1(k_i, k),$$

$$\dot{\bar{k}}_2 = \tilde{s}_2(k_i, k) \equiv \left(\frac{\left(A_i\overline{\psi}_i(k_i)k_i^{\alpha_i} + \delta k\right)\beta_s\xi_1 - A_i\beta_i\lambda_1\overline{\psi}_s(k_i)k_i^{\alpha_i}}{\lambda\beta_s\overline{N}_2} \right)\lambda_2\hat{N}(k_i, k) - \overline{\Omega}_2(k_i, k).$$

(A21)

Delete $\bar{k}_1(t)$ and $\bar{k}_2(t)$ from (A19) and (A21)

$$\overline{\Omega}_{1k_i}(k_i, k)\dot{k}_i + \overline{\Omega}_{1k}(k_i, k)\dot{k} = \tilde{s}_1(k_i, k),$$

$$\overline{\Omega}_{2k_i}(k_i, k)\dot{k}_i + \overline{\Omega}_{2k}(k_i, k)\dot{k} = \tilde{s}_2(k_i, k). \qquad \text{(A22)}$$

Solving (A13) with $\dot{k}_i(t)$ and $\dot{k}(t)$ as variables, we get

$$\dot{k}_i = \Lambda(k_i, k) \equiv \frac{\tilde{s}_1(k_i, k)\overline{\Omega}_{2k}(k_i, k) - \tilde{s}_2(k_i, k)\overline{\Omega}_{1k}(k_i, k)}{\overline{\Omega}_{1k_i}(k_i, k)\overline{\Omega}_{2k}(k_i, k) - \overline{\Omega}_{1k}(k_i, k)\overline{\Omega}_{2k_i}(k_i, k)},$$

$$\dot{k} = \Omega(k_i, k) \equiv \frac{\tilde{s}_2(k_i, k)\overline{\Omega}_{1k_i}(k_i, k) - \tilde{s}_1(k_i, k)\overline{\Omega}_{2k_i}(k_i, k)}{\overline{\Omega}_{1k_i}(k_i, k)\overline{\Omega}_{2k}(k_i, k) - \overline{\Omega}_{1k}(k_i, k)\overline{\Omega}_{2k_i}(k_i, k)}.$$

(A23)

The two differential equations contain two variables $k_i(t)$ and $k(t)$. Summarizing the above analysis, we confirmed the lemma.